Outline

Colored Proofs, Interpolation and Symbol Elimination

Interpolation

Theorem

Let A, B be closed formulas and let $A \vdash B$.

Then there exists a formula I such that

1. $A \vdash I$ *and* $I \vdash B$;
2. *every symbol of* I *occurs both in* A *and* B;

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Any formula I with this property is called an interpolant of A and B.

Essentially, an interpolant is a formula that is

1. intermediate in power between A and B;
2. Uses only common symbols of A and B.

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Interpolation has many uses in verification.

When we deal with refutations rather than proofs and have an unsatisfiable set $\{A, B\}$, it is convenient to use reverse interpolants of A and B, that is, a formula I such that

1. $A \vdash I$ and $\{I, B\}$ is unsatisfiable;
2. every symbol of I occurs both in A and B;

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- We have two formulas: A and B.
- Each symbol in A is either blue or green.
- Each symbol in B is either red or green.
- We know that $\vdash A \to B$.
- Our goal is to find a green formula I such that
  1. $\vdash A \to I$;
  2. $\vdash I \to B$.

Interpolation with Theories

- Theory T: any set of closed green formulas.
- $C_1, \ldots, C_n \vdash_T C$ denotes that the formula $C_1 \wedge \ldots \wedge C_1 \rightarrow C$ holds in all models of T.
- Interpreted symbols: symbols occurring in T.
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Theorem

Let A, B be formulas and let $A \vdash_T B$.

Then there exists a formula I such that

1. $A \vdash_T I$ *and* $I \vdash B$;
2. *every uninterpreted symbol of I occurs both in A and B;*
3. *every interpreted symbol of I occurs in B.*

Likewise, there exists a formula I such that

1. $A \vdash I$ *and* $I \vdash_T B$;
2. *every uninterpreted symbol of I occurs both in A and B;*
3. *every interpreted symbol of I occurs in A.*

Local Derivations

A derivation is called local (well-colored) if each inference in it

$$\frac{C_! \quad \cdots \quad C_n}{C}$$

either has no blue symbols or has no red symbols.
That is, one cannot mix blue and red in the same inference.

Local Derivations: Example

- A := $\forall x(x = a)$
- B := $c = b$
- Interpolant: $\forall x \forall y(x = y)$ (note: universally quantified!)
- Reverse interpolant: $\exists x \exists y(x \neq y)$

Local Derivations: Example

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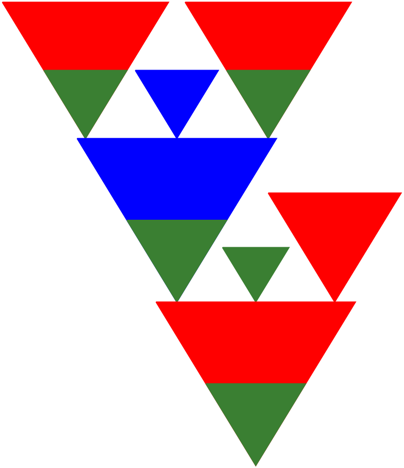
A local refutation in the superposition calculus:

$$\cfrac{\cfrac{x = a \quad y = a}{x = y} \qquad c \neq b}{\cfrac{y \neq b}{\bot}}$$

Shape of a local derivation



Symbol Eliminating Inference

- At least one of the premises is not green.
- The conclusion is green.

$$\dfrac{\boxed{\dfrac{x = a \quad y = a}{x = y}} \qquad c \neq b}{\boxed{\dfrac{y \neq b}{\bot}}}$$

Extracting Interpolants from Local Proofs

Theorem

Let Π be a local refutation. Then one can extract from Π in linear time a reverse interpolant I of A and B. This interpolant is ground if all formulas in Π are ground.

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What is remarkable in this theorem:

- No restriction on the calculus (only soundness required) – can be used with theories.
- Can generate interpolants in theories where no good interpolation algorithms exist.

Interpolation: Examples in Vampire

```
fof(fA,axiom, q(f(a)) & ~q(f(b)) ).
fof(fB,conjecture, ?[V]: V != c).
```

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```
% request to generate an interpolant
vampire(option,show_interpolant,on).
% symbol coloring
vampire(symbol,predicate,q,1,left).
vampire(symbol,function,f,1,left).
vampire(symbol,function,a,0,left).
vampire(symbol,function,b,0,left).
vampire(symbol,function,c,0,right).
% formula L
vampire(left_formula).
  fof(fA,axiom, q(f(a)) & ~q(f(b)) ).
vampire(end_formula).
% formula R
vampire(right_formula).
  fof(fB,conjecture, ?[V]: V != c).
vampire(end_formula).
```

Symbol Elimination

Colored proofs can also be used for an interesting application. Suppose that we have a set of formulas in some language and want to derive consequences of these formulas in a subset of this language.

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This technique was used in our experiments on automatic loop invariant generation.