Outline

Semantic Tableaux

Signed Formula

- **Signed formula**: an expression $A = b$, where A is a formula and b a boolean value.
- A signed formula $A = b$ is **true** in an interpretation I, denoted by $I \models A = b$, if $I(A) = b$.
- If $A = b$ is true in I, we also say that **I is a model of $A = b$**, or that **I satisfies $A = b$**.
- A signed formula is **satisfiable** if it has a model.

Note:

1. For every formula A and interpretation I **exactly one** of the signed formulas $A = 1$ and $A = 0$ is true in I.
2. A formula A is **satisfiable** if and only if so is the signed formula $A = 1$.

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How to find a model of a signed formula?

Example: $(A \to B) = 1$.

So $(A \to B) = 1$ if and only if $A = 0$ OR $B = 1$.

Likewise, $(A \to B) = 0$ if and only if $A = 1$ AND $B = 0$.

So we can use AND-OR trees to carry out case analysis.

Operation table for $\to$:

$\to$	$B = 1$	$B = 0$
$A = 1$	1	0
$A = 0$	1	1

$\to$	$B = 1$	$B = 0$
$A = 1$	1	0
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Operation table for $\rightarrow$:

$\rightarrow$	$B = 1$	$B = 0$
$A = 1$	1	0
$A = 0$	1	1

$\rightarrow$	$B = 1$	$B = 0$
$A = 1$	1	0
$A = 0$	1	1

How to find a model of a signed formula?

Example: $(A \to B) = 1$.

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Operation table for $\to$:

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$A = 1$	1	0
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Tableau

Tableau: a tree having signed formulas at nodes.

Tableau for a signed formula $A = b$ has $A = b$ as a root.

Alternatively, we can regard a tableau as a collection of **branches**; each branch is a set of signed formulas.

Notation for branches: $A_1 = b_1 \mid \ldots \mid A_n = b_n$.

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Branch Expansion Rules

$$
\begin{array}{rcl}
(A_1 \wedge \ldots \wedge A_n) = 0 & \rightsquigarrow & A_1 = 0 \mid \ldots \mid A_n = 0 \\
(A_1 \wedge \ldots \wedge A_n) = 1 & \rightsquigarrow & A_1 = 1, \ldots, A_n = 1 \\
\\
(A_1 \vee \ldots \vee A_n) = 0 & \rightsquigarrow & A_1 = 0, \ldots, A_n = 0 \\
(A_1 \vee \ldots \vee A_n) = 1 & \rightsquigarrow & A_1 = 1 \mid \ldots \mid A_n = 1 \\
\\
(A_1 \rightarrow A_2) = 0 & \rightsquigarrow & A_1 = 1, A_2 = 0 \\
(A_1 \rightarrow A_2) = 1 & \rightsquigarrow & A_1 = 0 \mid A_2 = 1 \\
\\
(\neg A_1) = 0 & \rightsquigarrow & A_1 = 1 \\
(\neg A_1) = 1 & \rightsquigarrow & A_1 = 0 \\
\\
(A_1 \leftrightarrow A_2) = 0 & \rightsquigarrow & A_1 = 0, A_2 = 1 \mid A_1 = 1, A_2 = 0 \\
(A_1 \leftrightarrow A_2) = 1 & \rightsquigarrow & A_1 = 0, A_2 = 0 \mid A_1 = 1, A_2 = 1
\end{array}
$$

Branch Closure Rules

These rules are introduced to mark when the set of signed formulas on a branch is unsatisfiable.

A branch is marked closed in any of the following cases:

- it contains both $p = 0$ and $p = 1$ for some atom p
- it contains $\top = 0$;
- it contains $\bot = 1$.

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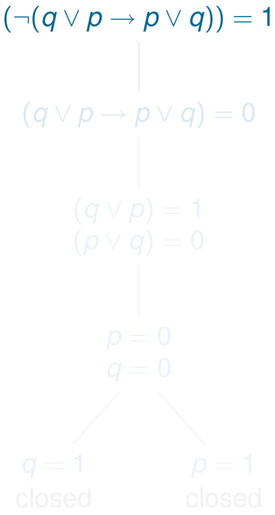
A Semantic Tableau



$(\neg(q \vee p \to p \vee q)) = 1$

$(q \vee p \to p \vee q) = 0$

$(q \vee p) = 1$
$(p \vee q) = 0$

$p = 0$
$q = 0$

$q = 1$
closed

$p = 1$
closed

$$\begin{aligned}
(A_1 \vee A_2) = 0 &\quad \rightsquigarrow \quad A_1 = 0, A_2 = 0 \\
(A_1 \vee A_2) = 1 &\quad \rightsquigarrow \quad A_1 = 1 \mid A_2 = 1 \\
(A_1 \to A_2) = 0 &\quad \rightsquigarrow \quad A_1 = 1, A_2 = 0 \\
(\neg A_1) = 1 &\quad \rightsquigarrow \quad A_1 = 0
\end{aligned}$$

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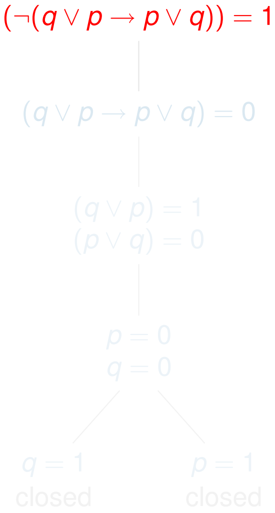

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$(q \vee p \to p \vee q) = 0$

$(q \vee p) = 1$
$(p \vee q) = 0$

$p = 0$
$q = 0$

$q = 1$
closed

$p = 1$
closed

$$\begin{array}{rcl}
(A_1 \vee A_2) = 0 & \rightsquigarrow & A_1 = 0, A_2 = 0 \\
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\end{array}$$

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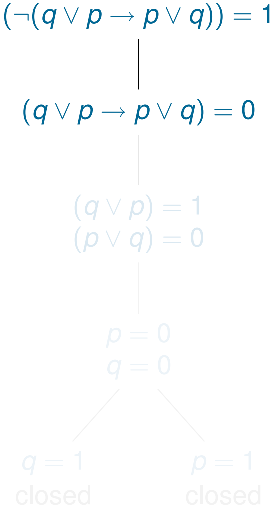

$(\neg(q \lor p \to p \lor q)) = 1$

$(q \lor p \to p \lor q) = 0$

$(q \lor p) = 1$
$(p \lor q) = 0$

$p = 0$
$q = 0$

$q = 1$ closed

$p = 1$ closed

$$
\begin{array}{rcl}
(A_1 \lor A_2) = 0 & \rightsquigarrow & A_1 = 0, A_2 = 0 \\
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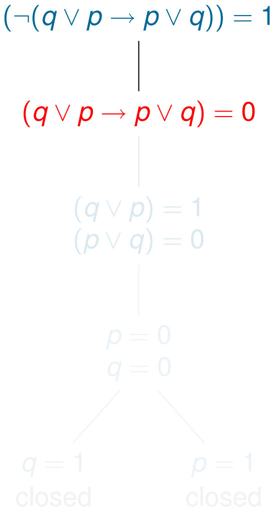

$(\neg(q \vee p \to p \vee q)) = 1$

$(q \vee p \to p \vee q) = 0$

$(q \vee p) = 1$

$(p \vee q) = 0$

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$q = 0$

$q = 1$

closed

$p = 1$

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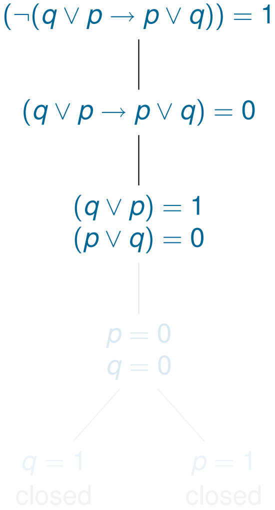

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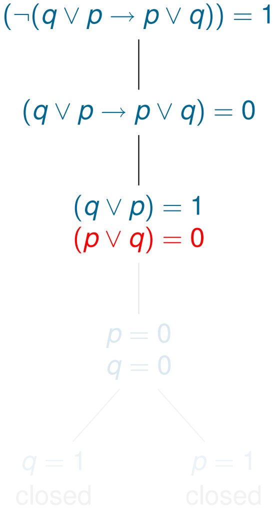

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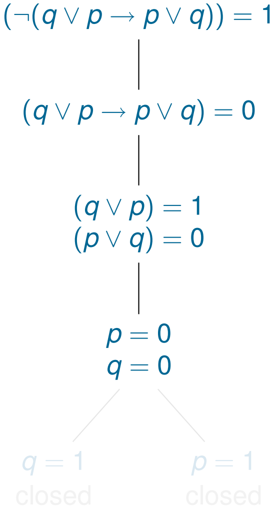

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$(q \vee p \rightarrow p \vee q) = 0$

$(q \vee p) = 1$
$(p \vee q) = 0$

$p = 0$
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$q = 1$
closed

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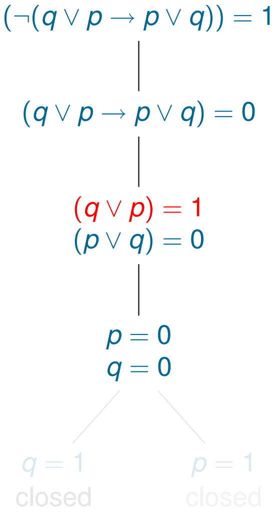

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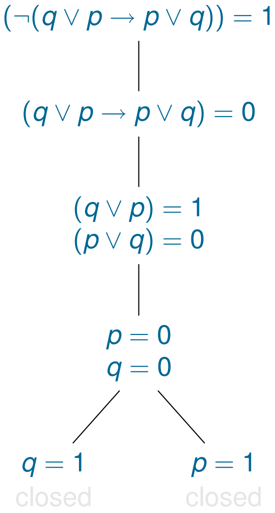

$(\neg(q \vee p \rightarrow p \vee q)) = 1$

$(q \vee p \rightarrow p \vee q) = 0$

$(q \vee p) = 1$
$(p \vee q) = 0$

$p = 0$
$q = 0$

$q = 1$ closed

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$$(A_1 \vee A_2) = 0 \quad \rightsquigarrow \quad A_1 = 0, A_2 = 0$$

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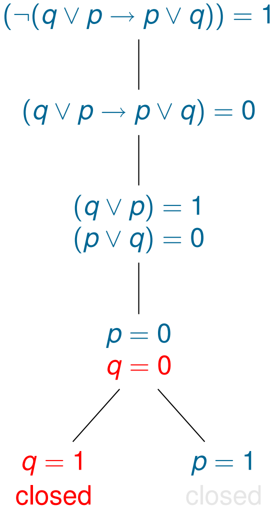

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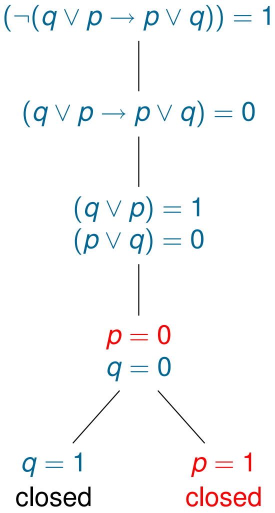

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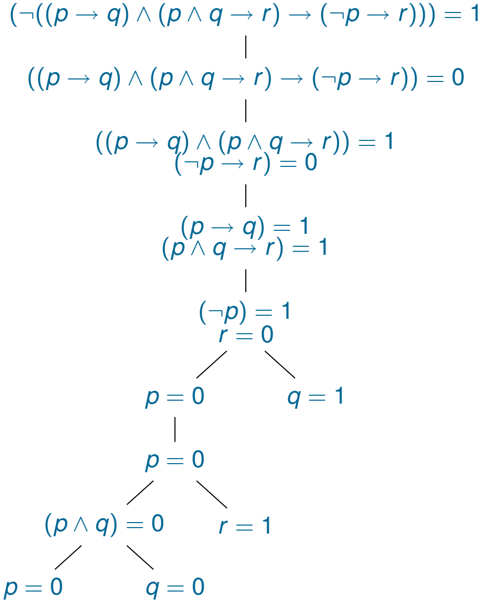
Finding Models Using Tableaux



$(\neg((p \to q) \land (p \land q \to r) \to (\neg p \to r))) = 1$

$((p \to q) \land (p \land q \to r) \to (\neg p \to r)) = 0$

$((p \to q) \land (p \land q \to r)) = 1$
$(\neg p \to r) = 0$

$(p \to q) = 1$
$(p \land q \to r) = 1$

$(\neg p) = 1$
$r = 0$

$p = 0$ $\quad$ $q = 1$

$p = 0$

$(p \land q) = 0$ $\quad$ $r = 1$

$p = 0$ $\quad$ $q = 0$

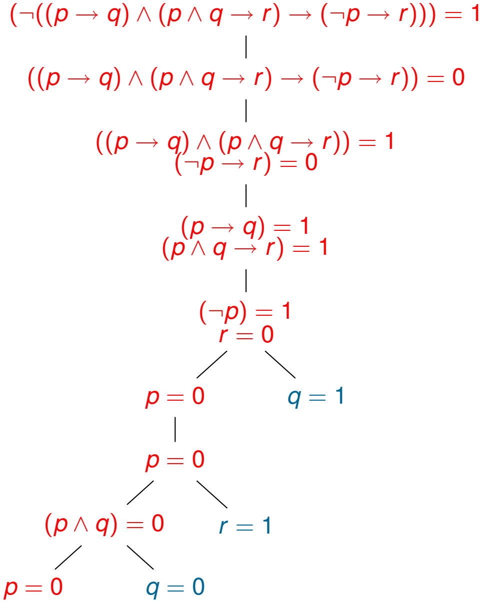
Build an open branch on which all rules have been applied.

Select signed atoms on this branch

They give us a model

$$\{r \mapsto 0, p \mapsto 0, q \mapsto \cdots\}$$

Finding Models Using Tableaux



$$(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (\neg p \rightarrow r))) = 1$$

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$$(p \rightarrow q) = 1$$
$$(p \wedge q \rightarrow r) = 1$$

$$(\neg p) = 1$$
$$r = 0$$

$$p = 0 \qquad q = 1$$

$$p = 0$$

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$$p = 0 \qquad q = 0$$

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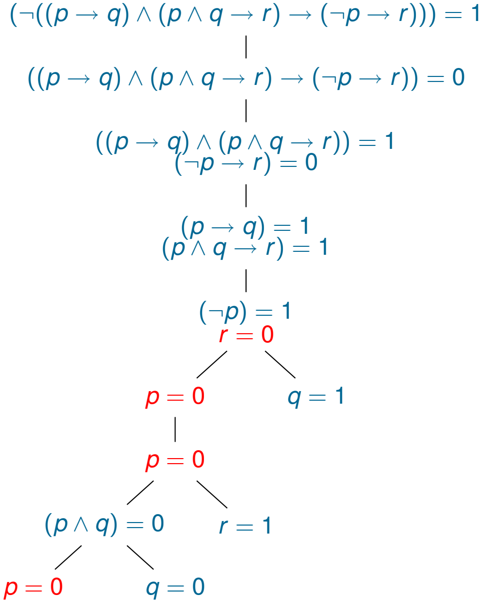

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$((p \to q) \land (p \land q \to r)) = 1$
$(\neg p \to r) = 0$

$(p \to q) = 1$
$(p \land q \to r) = 1$

$(\neg p) = 1$
$r = 0$

$p = 0$

$q = 1$

$p = 0$

$(p \land q) = 0$

$r = 1$

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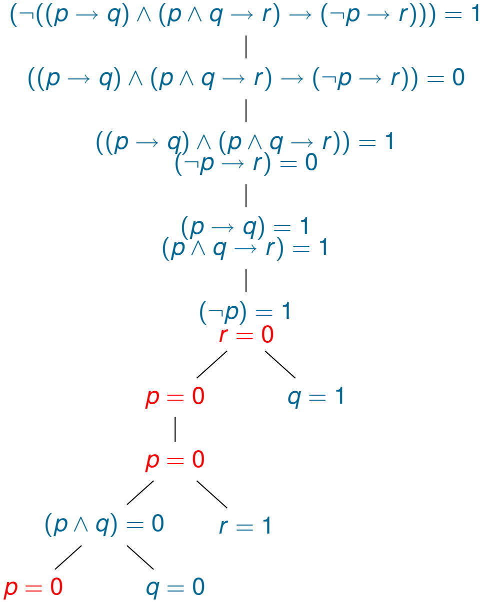

Build an open branch on which all rules have been applied.

Select signed atoms on this branch

They give us a model

$$\{r \mapsto 0, p \mapsto 0, q \mapsto \cdots\}$$

Checking Other Properties with Tableaux

A formula A is satisfiable iff a tableau for $A = 1$ contains a complete open branch (and iff every tableau for $A = 1$ contains a complete open branch).

A formula A is valid iff there is a closed a tableau for $A = 0$ (and iff every tableau for $A = 0$ is closed).

Formulas A and B are equivalent iff there is a closed tableau for $(A \leftrightarrow B) = 0$ (and iff every tableau for $(A \leftrightarrow B) = 0$ is closed).

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A formula A is valid iff there is a closed a tableau for $A = 0$ (and iff every tableau for $A = 0$ is closed).

Formulas A and B are equivalent iff there is a closed tableau for $(A \leftrightarrow B) = 0$ (and iff every tableau for $(A \leftrightarrow B) = 0$ is closed).

Checking Other Properties with Tableaux

A formula A is satisfiable iff a tableau for $A = 1$ contains a complete open branch (and iff every tableau for $A = 1$ contains a complete open branch).

A formula A is valid iff there is a closed a tableau for $A = 0$ (and iff every tableau for $A = 0$ is closed).

Formulas A and B are equivalent iff there is a closed tableau for $(A \leftrightarrow B) = 0$ (and iff every tableau for $(A \leftrightarrow B) = 0$ is closed).

End of Lecture 9

Slides for lecture 9 ended here . . .