

# Rectified formulas

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- ▶ no variable appears both free and bound in  $F$ ;
- ▶ for every variable  $p$ , the formula  $F$  contains at most one occurrence of quantifiers  $\exists p$ .

Any formula can be transformed into a rectified formula by renaming bound variables.

We can use the usual notation  $(F)_p^G$  for rectified formulas assuming that  $p$  occurs only free.

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# Rectification: Example

$$p \rightarrow \exists p(p \wedge \forall p(p \vee r \rightarrow \neg p)) \Rightarrow$$

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## Another problem

$\exists q(\neg p \leftrightarrow q)$ : there exists a truth value equivalent to  $\neg p$ .

Rename  $p$  into  $q$ .

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Suppose we want to substitute  $(F)_p^G$ .

Then we **require**: no free variable in  $G$  become bound in  $(F)_p^G$ .

In previous example  $\exists q(\neg p \leftrightarrow q)$ :

**Substitute  $p$  by  $q$** . ( $\exists q(\neg q \leftrightarrow q)$  does not satisfy above)

Uniform solution – renaming of bound variables

$\exists q(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$

Now we can substitute  $p$  by  $q$  obtaining  $\exists r(\neg q \leftrightarrow r)$

From now on we always assume that:

- ▶ formulas are **rectified**.
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# Prenex form

- ▶ **Quantifier-free formula:** no quantifiers (that is, propositional).
- ▶ **Prenex formula** has the form  $\exists_1 p_1 \dots \exists_n p_n G$ , where  $G$  is quantifier-free.
- ▶ **Outermost prefix of  $\exists_1 p_1 \dots \exists_n p_n G$ :** the longest subsequence  $\exists_1 p_1 \dots \exists_k p_k$  of  $\exists_1 p_1 \dots \exists_n p_n$  such that  $\exists_1 = \dots = \exists_k$ .
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# Prenexing rules

Prenexing rules:

$$\exists \forall p F_1 \times \dots \times F_n \Rightarrow \exists \forall p (F_1 \times \dots \times F_n)$$

$$\forall p F_1 \rightarrow F_2 \Rightarrow \exists p (F_1 \rightarrow F_2) \quad \exists p F_1 \rightarrow F_2 \Rightarrow \forall p (F_1 \rightarrow F_2)$$

$$F_1 \rightarrow \forall p F_2 \Rightarrow \forall p (F_1 \rightarrow F_2) \quad F_1 \rightarrow \exists p F_2 \Rightarrow \exists p (F_1 \rightarrow F_2)$$

$$\neg \forall p F \Rightarrow \exists p \neg F$$

$$\neg \exists p F \Rightarrow \forall p \neg F$$

# Prenexing. Example I

$$\begin{aligned} & \exists q(q \rightarrow p) \rightarrow \neg \forall r(r \rightarrow p) \vee p \Rightarrow \\ & \forall q((q \rightarrow p) \rightarrow \neg \forall r(r \rightarrow p) \vee p) \Rightarrow \\ & \forall q((q \rightarrow p) \rightarrow \exists r \neg(r \rightarrow p) \vee p) \Rightarrow \\ & \forall q((q \rightarrow p) \rightarrow \exists r(\neg(r \rightarrow p) \vee p)) \Rightarrow \\ & \forall q \exists r((q \rightarrow p) \rightarrow \neg(r \rightarrow p) \vee p). \end{aligned}$$

## Prenexing. Example II

$$\exists q(q \rightarrow p) \rightarrow \neg \forall r(r \rightarrow p) \vee p \Rightarrow$$

$$\exists q(q \rightarrow p) \rightarrow \exists r \neg(r \rightarrow p) \vee p \Rightarrow$$

$$\exists q(q \rightarrow p) \rightarrow \exists r(\neg(r \rightarrow p) \vee p) \Rightarrow$$

$$\exists r(\exists q(q \rightarrow p) \rightarrow \neg(r \rightarrow p) \vee p) \Rightarrow$$

$$\exists r \forall q((q \rightarrow p) \rightarrow \neg(r \rightarrow p) \vee p).$$

# What's next

Algorithms for satisfiability, validity of QBF:

- ▶ Splitting
- ▶ DLL

Reminder:

(i)  $F(p_1, \dots, p_n)$  is **satisfiable** iff  $\exists p_1 \dots \exists p_n F(p_1, \dots, p_n)$  is **true**

(ii)  $F(p_1, \dots, p_n)$  is **valid** iff  $\forall p_1 \dots \forall p_n F(p_1, \dots, p_n)$  is **true**

Algorithms will check **truth/falsity** of **closed** formulas.

# Splitting: Foundations

## Lemma

- ▶ A closed formula  $\forall p F$  is *true* if and only if the formulas  $F_p^\perp$  and  $F_p^\top$  are true.
- ▶ A closed formula  $\exists p F$  is *true* if and only if *at least one* of the formulas  $F_p^\perp$  or  $F_p^\top$  is true.

# Splitting

## Simplification rules for $\top$ :

$$\begin{aligned}\neg\top &\Rightarrow \perp \\ \top \wedge F_1 \wedge \dots \wedge F_n &\Rightarrow F_1 \wedge \dots \wedge F_n \\ \top \vee F_1 \vee \dots \vee F_n &\Rightarrow \top \\ F \rightarrow \top &\Rightarrow \top & \top \rightarrow F &\Rightarrow F \\ F \leftrightarrow \top &\Rightarrow F & \top \leftrightarrow F &\Rightarrow F \\ \forall p \top &\Rightarrow \top \\ \exists p \top &\Rightarrow \top\end{aligned}$$

## Simplification rules for $\perp$ :

$$\begin{aligned}\neg\perp &\Rightarrow \top \\ \perp \wedge F_1 \wedge \dots \wedge F_n &\Rightarrow \perp \\ \perp \vee F_1 \vee \dots \vee F_n &\Rightarrow F_1 \vee \dots \vee F_n \\ F \rightarrow \perp &\Rightarrow \neg F & \perp \rightarrow F &\Rightarrow \top \\ F \leftrightarrow \perp &\Rightarrow \neg F & \perp \leftrightarrow F &\Rightarrow \neg F \\ \forall p \perp &\Rightarrow \perp \\ \exists p \perp &\Rightarrow \perp\end{aligned}$$

# Splitting algorithm

**procedure** *splitting*( $F$ )

**input:** closed rectified prenex formula  $F$

**output:** 0 or 1

**parameters:** function *select\_signed\_atom*

**begin**

$F := \text{simplify}(F)$

**if**  $F = \perp$  **then return** 0

**if**  $F = \top$  **then return** 1

Let  $F$  have the form  $\exists \forall p_1 \dots \exists \forall p_k F_1$

$(p, b) := \text{select\_signed\_atom}(F)$

Let  $F'$  be obtained from  $F$  by deleting  $\exists \forall p$  from its outermost prefix

**if**  $b = 0$  **then**  $(G_1, G_2) := (\perp, \top)$

**else**  $(G_1, G_2) := (\top, \perp)$

**case** (*splitting*(( $F'$ ) $_{p^1}$ ),  $\exists \forall$ ) **of**

$(0, \forall) \Rightarrow$  **return** 0

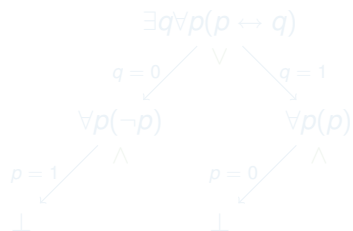
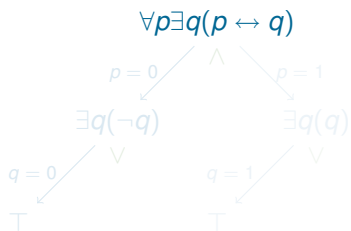
$(0, \exists) \Rightarrow$  **return** *splitting*(( $F'$ ) $_{p^2}$ )

$(1, \forall) \Rightarrow$  **return** *splitting*(( $F'$ ) $_{p^2}$ )

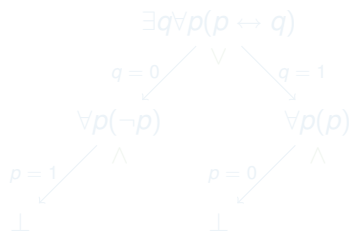
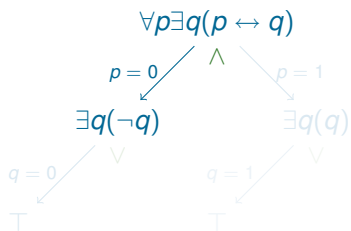
$(1, \exists) \Rightarrow$  **return** 1

**end**

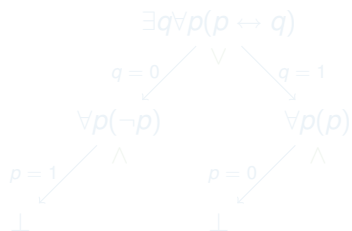
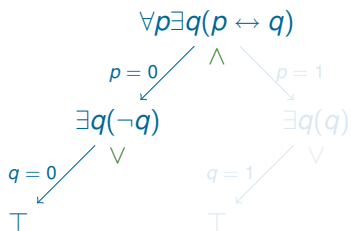
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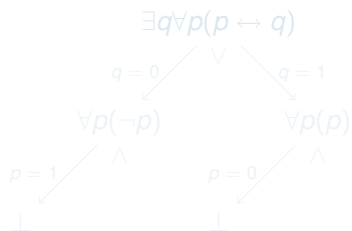
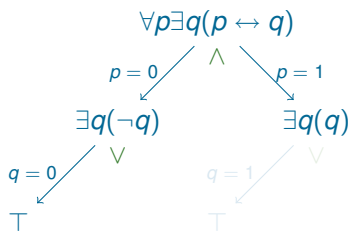
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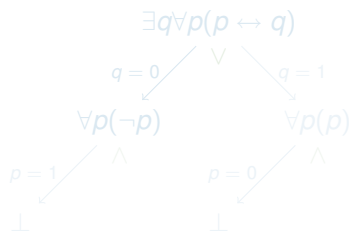
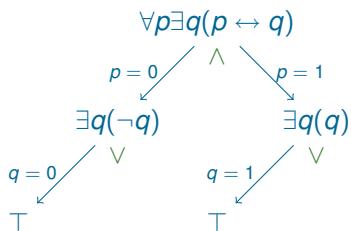
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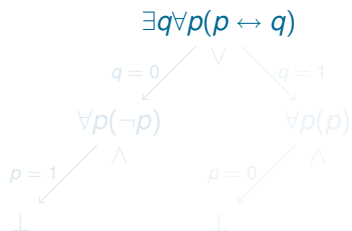
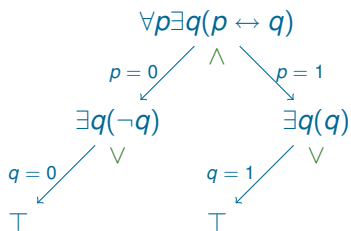
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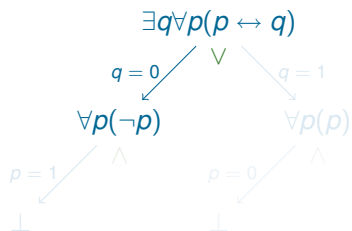
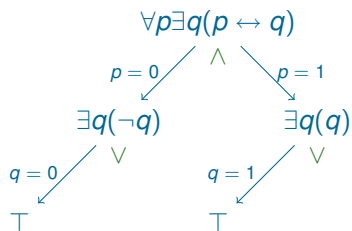
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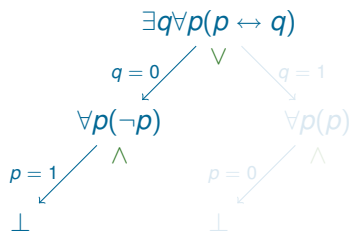
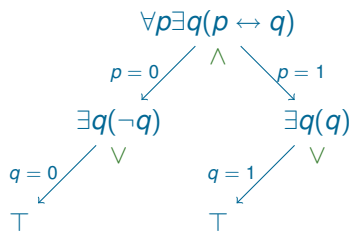
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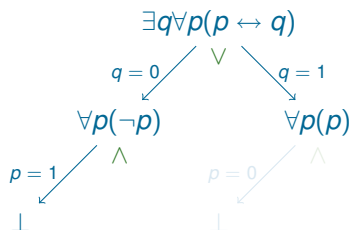
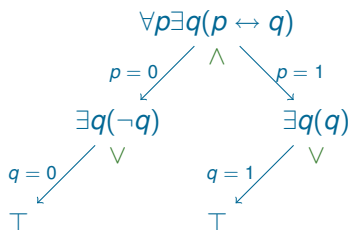
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