Symbolic Representation of Sets of States

Let $\mathbb{S} = (S, In, T, \mathcal{X}, dom, L)$ be a finite-state transition system. Then every formula F defines a set states:

$$\{s \mid s \models F\}.$$

We say that F (symbolically) represent this set of states.

Example

Let us represent the set of states in which the machine is ready to dispense a drink. In every such state, a drink should be available, the drink dispenser empty, and the coin slot contain enough coins. This can be expressed by:

$$(\mathrm{st\_coffee} \lor \mathrm{st\_beer}) \land \mathrm{disp} = \mathit{none} \land$$
$$((\mathrm{coins} = \mathit{1} \land \mathrm{st\_coffee}) \lor \mathrm{coins} = \mathit{2} \lor \mathrm{coins} = \mathit{3}).$$

Symbolic Representation of Transitions

- In addition to the set of propositional variables $\mathcal{X} = \{x_1, \ldots, x_n\}$, introduce a set of next state variables $\mathcal{X}' = \{x'_1, \ldots, x'_n\}$.
- Pairs of states as interpretations. For every variable $x \in \mathcal{X}$ define

$$
\begin{aligned}
(s, s')(x) &\stackrel{\text{def}}{\Leftrightarrow} s(x); \\
(s, s')(x') &\stackrel{\text{def}}{\Leftrightarrow} s'(x).
\end{aligned}
$$

- Symbolic representation. Formula F of variables $\mathcal{X} \cup \mathcal{X}'$ represents a transition t if $t = \{(s, s') \mid (s, s') \models F\}$.

Example

The transition *Recharge*:

$$\text{customer} = \textit{none} \wedge \text{st\_coffee}' \wedge \text{st\_beer}'.$$

But this formula includes describes a very strange transition after which, for example

- coins may appear in and disappear from the slot;
- dfrinks may appear in and disappear from the dispenser.
- ...

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Frame problem

One has to express explicitly, maybe for a large number of state variables, that the values of these variables do not change after a transition. For example,

$$\begin{array}{l}(\text{coins} = 0 \leftrightarrow \text{coins}' = 0) \wedge \\ (\text{coins} = 1 \leftrightarrow \text{coins}' = 1) \wedge \\ (\text{coins} = 2 \leftrightarrow \text{coins}' = 2) \wedge \\ (\text{coins} = 3 \leftrightarrow \text{coins}' = 3).\end{array}$$

This frame problem arises in artificial intelligence, knowledge representation, and reasoning about actions.

Notation for the frame formula

Abbreviations (we assume $dom(x) = dom(y)$):

$$x \neq v \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \neg(x = v)$$

$$x = y \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \bigwedge_{v \in dom(x)} (x = v \leftrightarrow y = v).$$

Let $\mathbb{S}$ be a transition system and $\{x_1, \ldots, x_n\} \subseteq \mathcal{X}$ be a set of state variables of $\mathcal{L}(\mathbb{S})$. Define

$$only(x_1, \ldots, x_n) \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \bigwedge_{y \in \mathcal{X} \setminus \{x_1, \ldots, x_n\}} y = y'.$$

This formula expresses that $x_1, \ldots, x_n$ are the only variables whose values can be changed by the transition.

Preconditions and postconditions

When we represent a transition symbolically using a formula F of variables $\mathcal{X} \cup \mathcal{X}'$, the formula F is usually represented as the conjunction $F_1 \land F_2$ of two formulas:

1. F_1 expresses some conditions on the variables $\mathcal{X}$ which are necessary to execute the transition (precondition);
2. F_2 expresses some conditions relating variables in $\mathcal{X}$ to those in $\mathcal{X}'$, i.e., conditions which show how the values of the variables after the transition relate to their values before the transition (postcondition).

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Transitions

$Recharge \overset{\text{def}}{\Leftrightarrow}$ $\text{customer} = none \wedge \text{st\_coffee}' \wedge \text{st\_beer}' \wedge$ $only(\text{st\_coffee}, \text{st\_beer})$.

$Customer\_arrives \overset{\text{def}}{\Leftrightarrow}$ $\text{customer} = none \wedge \text{customer}' \neq none \wedge$ $only(\text{customer})$

$Customer\_leaves \overset{\text{def}}{\Leftrightarrow}$ $\text{customer} \neq none \wedge \text{customer}' = none \wedge$ $only(\text{customer})$.

$Coin\_insert \overset{\text{def}}{\Leftrightarrow}$ $\text{customer} \neq none \wedge \text{coins} \neq 3 \wedge$
$(\text{coins} = 0 \rightarrow \text{coins}' = 1) \wedge$
$(\text{coins} = 1 \rightarrow \text{coins}' = 2) \wedge$
$(\text{coins} = 2 \rightarrow \text{coins}' = 3) \wedge$
$only(\text{coins})$.

Transitions

$$\mathit{Recharge} \overset{\text{def}}{\Leftrightarrow} \text{customer} = \mathit{none} \wedge \text{st\_coffee}' \wedge \text{st\_beer}' \wedge \mathit{only}(\text{st\_coffee}, \text{st\_beer}).$$

$$\mathit{Customer\_arrives} \overset{\text{def}}{\Leftrightarrow} \text{customer} = \mathit{none} \wedge \text{customer}' \neq \mathit{none} \wedge \mathit{only}(\text{customer})$$

$$\mathit{Customer\_leaves} \overset{\text{def}}{\Leftrightarrow} \text{customer} \neq \mathit{none} \wedge \text{customer}' = \mathit{none} \wedge \mathit{only}(\text{customer}).$$

$$\mathit{Coin\_insert} \overset{\text{def}}{\Leftrightarrow} \text{customer} \neq \mathit{none} \wedge \text{coins} \neq \mathit{3} \wedge (\text{coins} = \mathit{0} \rightarrow \text{coins}' = \mathit{1}) \wedge (\text{coins} = \mathit{1} \rightarrow \text{coins}' = \mathit{2}) \wedge (\text{coins} = \mathit{2} \rightarrow \text{coins}' = \mathit{3}) \wedge \mathit{only}(\text{coins}).$$

Transitions

$$\mathit{Recharge} \overset{\text{def}}{\Leftrightarrow} \text{customer} = \mathit{none} \land \text{st\_coffee}' \land \text{st\_beer}' \land \mathit{only}(\text{st\_coffee}, \text{st\_beer}).$$

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Transitions

$$\mathit{Dispense\_beer} \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} \text{customer} = \mathit{student} \land \text{st\_beer} \land \\ \text{disp} = \mathit{none} \land (\text{coins} = \mathit{2} \lor \text{coins} = \mathit{3}) \land \\ \text{disp}' = \mathit{beer} \land \\ (\text{coins} = \mathit{2} \rightarrow \text{coins}' = \mathit{0}) \land \\ (\text{coins} = \mathit{3} \rightarrow \text{coins}' = \mathit{1}) \land \\ \mathit{only}(\text{st\_beer}, \text{disp}, \text{coins}). \end{array}$$

$$\mathit{Dispense\_coffee} \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} \text{customer} = \mathit{prof} \land \text{st\_coffee} \land \\ \text{disp} = \mathit{none} \land \text{coins} \neq \mathit{0} \land \\ \text{disp}' = \mathit{coffee} \land \\ (\text{coins} = \mathit{1} \rightarrow \text{coins}' = \mathit{0}) \land \\ (\text{coins} = \mathit{2} \rightarrow \text{coins}' = \mathit{1}) \land \\ (\text{coins} = \mathit{3} \rightarrow \text{coins}' = \mathit{2}) \land \\ \mathit{only}(\text{st\_coffee}, \text{disp}, \text{coins}). \end{array}$$

$$\mathit{Take\_drink} \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} \text{customer} \neq \mathit{none} \land \text{disp} \neq \mathit{none} \land \\ \text{disp}' = \mathit{none} \land \\ \mathit{only}(\text{disp}). \end{array}$$

Transitions

$$
\begin{aligned}
\textit{Dispense\_beer} \;&\overset{\text{def}}{\Leftrightarrow}\; \text{customer} = \textit{student} \wedge \text{st\_beer} \wedge \\
&\quad \text{disp} = \textit{none} \wedge (\text{coins} = \textit{2} \vee \text{coins} = \textit{3}) \wedge \\
&\quad \text{disp}' = \textit{beer} \wedge \\
&\quad (\text{coins} = \textit{2} \rightarrow \text{coins}' = \textit{0}) \wedge \\
&\quad (\text{coins} = \textit{3} \rightarrow \text{coins}' = \textit{1}) \wedge \\
&\quad \textit{only}(\text{st\_beer}, \text{disp}, \text{coins}).
\end{aligned}
$$

$$
\begin{aligned}
\textit{Dispense\_coffee} \;&\overset{\text{def}}{\Leftrightarrow}\; \text{customer} = \textit{prof} \wedge \text{st\_coffee} \wedge \\
&\quad \text{disp} = \textit{none} \wedge \text{coins} \neq \textit{0} \wedge \\
&\quad \text{disp}' = \textit{coffee} \wedge \\
&\quad (\text{coins} = \textit{1} \rightarrow \text{coins}' = \textit{0}) \wedge \\
&\quad (\text{coins} = \textit{2} \rightarrow \text{coins}' = \textit{1}) \wedge \\
&\quad (\text{coins} = \textit{3} \rightarrow \text{coins}' = \textit{2}) \wedge \\
&\quad \textit{only}(\text{st\_coffee}, \text{disp}, \text{coins}).
\end{aligned}
$$

$$
\begin{aligned}
\textit{Take\_drink} \;&\overset{\text{def}}{\Leftrightarrow}\; \text{customer} \neq \textit{none} \wedge \text{disp} \neq \textit{none} \wedge \\
&\quad \text{disp}' = \textit{none} \wedge \\
&\quad \textit{only}(\text{disp}).
\end{aligned}
$$

Transitions

$$
\begin{aligned}
\textit{Dispense\_beer} \overset{\text{def}}{\Leftrightarrow}\ & \text{customer} = \textit{student} \wedge \text{st\_beer} \wedge \\
& \text{disp} = \textit{none} \wedge (\text{coins} = \textit{2} \vee \text{coins} = \textit{3}) \wedge \\
& \text{disp}' = \textit{beer} \wedge \\
& (\text{coins} = \textit{2} \rightarrow \text{coins}' = \textit{0}) \wedge \\
& (\text{coins} = \textit{3} \rightarrow \text{coins}' = \textit{1}) \wedge \\
& \textit{only}(\text{st\_beer}, \text{disp}, \text{coins}).
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$$

$$
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\textit{Dispense\_coffee} \overset{\text{def}}{\Leftrightarrow}\ & \text{customer} = \textit{prof} \wedge \text{st\_coffee} \wedge \\
& \text{disp} = \textit{none} \wedge \text{coins} \neq \textit{0} \wedge \\
& \text{disp}' = \textit{coffee} \wedge \\
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& \textit{only}(\text{st\_coffee}, \text{disp}, \text{coins}).
\end{aligned}
$$

$$
\begin{aligned}
\textit{Take\_drink} \overset{\text{def}}{\Leftrightarrow}\ & \text{customer} \neq \textit{none} \wedge \text{disp} \neq \textit{none} \wedge \\
& \text{disp}' = \textit{none} \wedge \\
& \textit{only}(\text{disp}).
\end{aligned}
$$

Transitions

Model checkers often use a convention that the variables that can change are those variables x such that x' occurs in the problem. Under this convention we can remove $only(\ldots)$ from all transitions and change $Dispense\_beer$ and $Dispense\_coffee$ as follows:

$$
\begin{array}{lll}
Dispense\_beer & \stackrel{\text{def}}{\Leftrightarrow} & \text{customer} = student \land \text{st\_beer} \land \\
 & & \text{disp} = none \land (\text{coins} = 2 \lor \text{coins} = 3) \land \\
 & & \text{disp}' = beer \land \\
 & & (\text{coins} = 2 \rightarrow \text{coins}' = 0) \land (\text{coins} = 3 \rightarrow \text{coins}' = 1) \land \\
 & & \text{st\_beer}' = \text{st\_beer}'. \\
Dispense\_coffee & \stackrel{\text{def}}{\Leftrightarrow} & \text{customer} = prof \land \text{st\_coffee} \land \\
 & & \text{disp} = none \land \text{coins} \neq 0 \land \\
 & & \text{disp}' = coffee \land \\
 & & (\text{coins} = 1 \rightarrow \text{coins}' = 0) \land (\text{coins} = 2 \rightarrow \text{coins}' = 1) \land \\
 & & (\text{coins} = 3 \rightarrow \text{coins}' = 2) \land \\
 & & \text{st\_coffee}' = \text{st\_coffee}'.
\end{array}
$$

Temporal properties of transition systems

1. There is no state in which professor and student are both customers.
2. Students never drink coffee.
3. The machine cannot dispense drinks forever without recharging.

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