

Interpreting formulas

Extend I to all formulas:

1. $I(\top) = 1$ and $I(\perp) = 0$.
2. $I(A_1 \wedge \dots \wedge A_n) = 1$ if and only if $I(A_i) = 1$ for all i .
3. $I(A_1 \vee \dots \vee A_n) = 1$ if and only if $I(A_i) = 1$ for some i .
4. $I(\neg A) = 1$ if and only if $I(A) = 0$.
5. $I(A_1 \rightarrow A_2) = 1$ if and only if $I(A_1) = 0$ or $I(A_2) = 1$.
6. $I(A_1 \leftrightarrow A_2) = 1$ if and only if $I(A_1) = I(A_2)$.

Operation tables

$I(A_1 \vee A_2) = 1$ if and only if $I(A_1) = 1$ or $I(A_2) = 1$.

$I(A_1 \leftrightarrow A_2) = 1$ if and only if $I(A_1) = I(A_2)$.

\wedge	1	0	\vee	1	0	\neg	
1	1	0	1	1	1	1	0
0	0	0	0	1	0	0	1
\rightarrow	1	0	\leftrightarrow	1	0		
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Satisfiability, validity

- ▶ If $I(A) = 1$, then we say that the formula A is **true** in I and that I **satisfies** A and that I **is a model of** A , denoted by $I \models A$.
- ▶ If $I(A) = 0$, then we say that the formula A is **false** in I .
- ▶ A is **satisfiable** (**valid**) if it is true in some (every) interpretation.
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Examples

$A \rightarrow A$ and $A \vee \neg A$ are **valid** for all formulas A .

Evidently, every valid formula is also satisfiable.

$A \wedge \neg A$ is unsatisfiable.

Formula p , where p is a boolean variable, is satisfiable but not valid.

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Examples: equivalences

For all formulas A and B , the following equivalences hold.

$$A \rightarrow \perp \equiv \neg A; \quad (1)$$

$$\top \rightarrow A \equiv A; \quad (2)$$

$$A \rightarrow B \equiv \neg(A \wedge \neg B); \quad (3)$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B); \quad (4)$$

$$A \vee B \equiv \neg A \rightarrow B. \quad (5)$$

Connections between these notions

1. A formula A is **valid** if and only if $\neg A$ is **unsatisfiable**.
2. A formula A is **satisfiable** if and only if $\neg A$ is **not valid**.
3. A formula A is **valid** if and only if A is **equivalent** to \top .
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How to evaluate a formula?

Let's evaluate the formula

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$$

in the interpretation

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}.$$

Evaluating a formula

formula				value
$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				1
$p \rightarrow r$				1
$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				0
$p \wedge q \rightarrow r$				1
$p \rightarrow q$				0
$p \wedge q$				0
p	p	p		1
q	q			0
		r	r	1

Evaluating a formula

formula	value
$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	1
$p \rightarrow r$	1
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$p \rightarrow q$	0
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p	1
q	0
r	1

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$p \wedge q \rightarrow r$				1
$p \rightarrow q$				0
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p	p	p		1
	q	q		0
		r	r	1

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q	q	q	q	0
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Equivalent replacement

We denote by $A[B]$ a formula A with a fixed occurrence of a subformula B . If we use this notation we can also write $A[B']$ to denote the formula obtained from A by replacing this occurrence of B by B' .

Lemma (Equivalent Replacement)

Let I be an interpretation and $I \models A_1 \leftrightarrow A_2$. Then $I \models B[A_1] \leftrightarrow B[A_2]$.

Theorem (Equivalent Replacement)

Let $A_1 \equiv A_2$. Then $B[A_1] \equiv B[A_2]$.

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A purely syntactic algorithm

If $I \models p$, then $I \models p \leftrightarrow \top$;

If $I \not\models p$, then $I \models p \leftrightarrow \perp$;

Since we can replace equivalent formulas by equivalent, we can replace every atom p by either \top or \perp , depending on the value of p in I .

Rewrite rules for evaluating a formula

Suppose that we have a formula consisting only of \perp and \top .
One can note that every formula of this form different from \perp and \top can be “simplified” to a smaller equivalent formula.

For example, every formula of the form $A \rightarrow \top$ is equivalent to a simpler formula \top .

This simplification process can be formalised as a system of rewrite rules:

$$\begin{array}{l} \top \wedge \dots \wedge \top \Rightarrow \top \\ \perp \wedge A_1 \wedge \dots \wedge A_n \Rightarrow \perp \end{array}$$

$$\begin{array}{l} \top \vee A_1 \vee \dots \vee A_n \Rightarrow \top \\ \perp \vee \dots \vee \perp \Rightarrow \perp \end{array}$$

$$\begin{array}{l} \neg \top \Rightarrow \perp \\ \neg \perp \Rightarrow \top \end{array}$$

$$\begin{array}{l} A \rightarrow \top \Rightarrow \top \\ \perp \rightarrow A \Rightarrow \top \\ \top \rightarrow \perp \Rightarrow \perp \end{array}$$

$$\begin{array}{l} \top \leftrightarrow \top \Rightarrow \top \\ \top \leftrightarrow \perp \Rightarrow \perp \\ \perp \leftrightarrow \top \Rightarrow \perp \\ \perp \leftrightarrow \perp \Rightarrow \top \end{array}$$

Read about the rewrite rules in Chapter 2.

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